

THE FULL EXCEPTIONAL COLLECTIONS OF CATEGORICAL RESOLUTIONS OF CURVES

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ABSTRACT. This paper gives a complete answer of the following question: which (singular, projective) curves have a categorical resolution of singularities which admits a full exceptional collection? We prove that such full exceptional collection exists if and only if the geometric genus of the curve equals to 0. Moreover we can also prove that a curve with geometric genus equal or greater than 1 cannot have a categorical resolution of singularities which has a tilting object. The proofs of both results are given by a careful study of the Grothendieck group and the Picard group of that curve.

Keywords: categorical resolution, singular curve, K-theory, exceptional collection, tilting object

MSC: 14H20, 14F05, 18E30, 18F30

1. INTRODUCTION

For a triangulated category \mathcal{C} , having a full exceptional collection is a very good property. Recall that the definition of full exceptional collection is as follows.

Definition 1.1. A full exceptional collection of a triangulated category \mathcal{C} is a collection $\{A_1 \dots A_n\}$ of objects such that

- (1) for all i one has $\mathrm{Hom}_{\mathcal{C}}(A_i, A_i) = k$ and $\mathrm{Hom}_{\mathcal{C}}(A_i, A_i[l]) = 0$ for all $l \neq 0$;
- (2) for all $1 \leq i < j \leq n$ one has $\mathrm{Hom}_{\mathcal{C}}(A_j, A_i[l]) = 0$ for all $l \in \mathbb{Z}$;
- (3) the smallest triangulated subcategory of \mathcal{C} containing A_1, \dots, A_n coincides with \mathcal{C} .

However it is not very common that a triangulated category \mathcal{C} has a full exceptional collection. In algebraic geometry, it is well-known that for a smooth projective curve X over an algebraically closed field k , its bounded derived category of coherent sheaves $D^b(\mathrm{coh}(X))$ has a full exceptional collection if and only if the genus of X equals to 0.

Moreover for a singular projective curve X and a (geometric) resolution of singularities $\tilde{X} \rightarrow X$, the geometric genus of \tilde{X} and X are equal, hence it is clear that $D^b(\mathrm{coh}(\tilde{X}))$ has a full exceptional collection if and only if the geometric genus of X equals to 0.

In this paper we would like to consider the categorical resolution of X , which is introduced in [4].

Definition 1.2. [[4] Definition 3.2 or [5] Definition 1.3] A categorical resolution of a scheme X is a smooth, cocomplete, compactly generated, triangulated category \mathcal{T} with an adjoint pair of triangulated functors

$$\pi^* : D(X) \rightarrow \mathcal{T} \text{ and } \pi_* : \mathcal{T} \rightarrow D(X)$$

such that

- (1) $\pi_* \circ \pi^* = id$;
- (2) both π_* and π^* commute with arbitrary direct sums;
- (3) $\pi_*(\mathcal{T}^c) \subset D^b(\mathrm{coh}(X))$ where \mathcal{T}^c denotes the full subcategory of \mathcal{T} which consists of compact objects.

Remark 1. The first property implies that π^* is fully faithful and the second property implies that $\pi^*(D^{\mathrm{perf}}(X)) \subset \mathcal{T}^c$.

Remark 2. The categorical resolution of X is not necessarily unique.

Remark 3. In this paper we will not discuss further on the smoothness of a triangulated category and the interested readers may refer to [5] Section 1. Moreover, the main result in this paper does not depend on the smoothness, see Corollary 3.6 and Corollary 4.8 below.

We are interested in the question that when does \mathcal{T}^c have a full exceptional collection. If X is an projective curve of geometric genus $g = 0$, it can be deduced from the construction in [5] that there exists a categorical resolution $(\mathcal{T}, \pi^*, \pi_*)$ of X such that \mathcal{T}^c has a full exceptional collection. See Proposition 4.1 below.

The main result of this paper is the following theorem, which rules out the possibility for any categorical resolution of a curve with geometric genus $g \geq 1$ has a full exceptional collection.

Theorem 1.1. *[See Theorem 4.9 below] Let X be a projective curve over an algebraically closed field k . Let $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of X . If the geometric genus of X is ≥ 1 , then \mathcal{T}^c cannot have a full exceptional collection.*

In other words, X has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of X equals to 0.

Remark 4. In a recent paper [1] a result which is related to the above claim has been proved. Actually it has been proved that if X is a reduced rational curve, then there exists a categorical resolution $(\mathcal{T}, \pi^*, \pi_*)$ of X such that \mathcal{T}^c has a tilting object, which in general does not come from an exceptional collection. See [1] Theorem 7.4.

Recall that the definition of tilting object is given as follows.

Definition 1.3. Let \mathcal{C} be a triangulated category. A tilting object is an object L of \mathcal{C} which satisfies the following properties.

- (1) L is a compact object of \mathcal{C} ;
- (2) $\text{Hom}_{\mathcal{C}}(L, L[i]) = 0$ for any non-zero integer i ;
- (3) the smallest thick triangulated subcategory of \mathcal{C} which contains L is \mathcal{C} itself.

For a tilting object let $\Lambda = \text{End}_{\mathcal{C}}(L)$. Then it can be shown that we have equivalence of triangulated categories

$$\mathcal{C} \cong \text{D}^b(\Lambda - \text{mod})$$

where $\text{D}^b(\Lambda - \text{mod})$ is the derived category of bounded complexes of finitely generated Λ -modules.

Actually we can also prove a related result in the $g \geq 1$ case. (thanks to Igor Burban for pointing it out)

Theorem 1.2. *[See Theorem 4.10 below] Let X be a projective curve over an algebraically closed field k of geometric genus ≥ 1 . Let $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of X . Then \mathcal{T}^c cannot have a tilting object, moreover there cannot be a finite dimensional k -algebra Λ of finite global dimension such that*

$$\mathcal{T}^c \cong \text{D}^b(\Lambda - \text{mod})$$

The proofs of both theorems depend on a careful study of various Grothendieck groups of X . In particular we will investigate the natural map $K_0(\text{D}^{\text{perf}}(X)) \rightarrow K_0(\text{D}^b(\text{coh}(X)))$ and show that if $g \geq 1$ then the image is not finitely generated, of which Theorem 1.1 and 1.2 will be a direct consequence.

2. SOME GENERALITIES ON K-THEORY AND THE PICARD GROUP

In this section we quickly review the K-theory and the Picard group of schemes. For reference see [9] Chapter II.

Let \mathcal{A} be an abelian category (or more generally an exact category). The Grothendieck group $K_0(\mathcal{A})$ is defined as an abelian group with generators $[A]$ for each isomorphism class of objects A in \mathcal{A} and subjects to the relation that

$$[A_2] = [A_1] + [A_3]$$

for any short exact sequence $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ in \mathcal{A} .

Similarly let \mathcal{C} be a triangulated category. The Grothendieck group $K_0(\mathcal{C})$ is defined as an abelian group with generators $[C]$ for each isomorphism class of objects C in \mathcal{C} and subjects to the relation that

$$[C_2] = [C_1] + [C_3]$$

for any exact triangle $C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_1[1]$ in \mathcal{C} .

Proposition 2.1. *If a triangulated category \mathcal{C} has a full exceptional collection $\{A_1 \dots A_n\}$, then the Grothendieck group of \mathcal{C} , $K_0(\mathcal{C})$, is isomorphic to \mathbb{Z}^n .*

Proof. It is an immediate consequence of Definition 1.1. □

Definition 2.1. Let X be an Noetherian scheme, follow the standard notation (see for example [8] Section 5.6 or [9] Chapter II) we denote the Grothendieck group of $D^{\text{perf}}(X)$ by $K_0(X)$ and the Grothendieck group of $D^b(\text{coh}(X))$ by $G_0(X)$.

Notice that in some literatures, say [3] Exposé IV or [7], $K_0(X)$ is denoted by $K^0(X)$ and $G_0(X)$ is denoted by $K_0(X)$. Nevertheless in this paper we will use the previous notation.

Remark 5. In the literature people also define $K_0^{\text{naive}}(X)$ to be the Grothendieck group of the exact category $VB(X)$ and $G_0^{\text{naive}}(X)$ to be the Grothendieck group of the abelian category $\text{coh}(X)$.

Nevertheless $G_0^{\text{naive}}(X)$ is isomorphic to $G_0(X)$ for any Noetherian scheme X ([3], Exposé IV, 2.4) and $K_0^{\text{naive}}(X)$ is isomorphic to $K_0(X)$ for any quasi-projective scheme X ([3], Exposé IV, 2.9). Since we always work with quasi-projective schemes in this paper, we can identify $G_0^{\text{naive}}(X)$ and $G_0(X)$ as well as $K_0^{\text{naive}}(X)$ and $K_0(X)$.

Definition 2.2. Let X be a Noetherian scheme. The inclusion $D^{\text{perf}}(X) \hookrightarrow D^b(\text{coh}(X))$ gives a group homomorphism

$$c : K_0(X) \rightarrow G_0(X)$$

which is called the Cartan homomorphism.

Proposition 2.2. For a Noetherian scheme X , the tensor product gives $K_0(X)$ a ring structure and $G_0(X)$ a $K_0(X)$ -module structure. Moreover, the Cartan homomorphism $c : K_0(X) \rightarrow G_0(X)$ is a morphism of $K_0(X)$ -modules.

Proof. See [7] 1.5 and 1.6. □

Proposition 2.3. If X is a regular Noetherian scheme, then the Cartan homomorphism is an isomorphism, i.e. we have

$$c : K_0(X) \xrightarrow{\cong} G_0(X)$$

Proof. See [9] Chapter II Theorem 8.2. □

Smooth schemes are regular hence the Cartan homomorphism is an isomorphism for any smooth scheme.

Remark 6. For general X the Cartan homomorphism is not an isomorphism, actually it is not even injective in general.

Next we talk about the functorial properties of K_0 and G_0 , which are more involved. First we have the following definition.

Definition 2.3. Let $f : X \rightarrow Y$ be a morphism of schemes, then the derived pull-back Lf^* functor induces the map

$$f^* : K_0(Y) \rightarrow K_0(X).$$

See [3] Exposé IV, 2.7.

If $f : X \rightarrow Y$ is a flat morphism between Noetherian schemes, or more generally f is of finite Tor-dimension. Then Lf^* is a functor $D^b(\text{coh}(Y)) \rightarrow D^b(\text{coh}(X))$ and induces the map

$$f^* : G_0(Y) \rightarrow G_0(X).$$

See [3] Exposé IV, 2.12.

We can also define the push-forward map for $G_0(-)$ for proper morphisms.

Definition 2.4. Let $f : X \rightarrow Y$ be a proper morphism of Noetherian schemes, then the derived push-forward functor Rf_* induces the map

$$f_* : G_0(X) \rightarrow G_0(Y).$$

We will also need some results on the relationship between the Grothendieck group and the Picard group. Let $\text{Pic}(X)$ denote the Picard group of X and we have the following proposition.

Proposition 2.4. There is a determinant map

$$\det : K_0(X) \rightarrow \text{Pic}(X)$$

which is a surjective group homomorphism. Moreover, the determinant map commutes with the restriction map, i.e. we have the following commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\det} & \text{Pic}(X) \\ \downarrow r & & \downarrow r \\ K_0(U) & \xrightarrow{\det} & \text{Pic}(U) \end{array}$$

Proof. For an n -dimensional vector bundle \mathcal{E} we can take its determinant line bundle, i.e. the top exterior power $\wedge^n \mathcal{E}$ and we call it $\det(\mathcal{E})$. Moreover, for a short exact sequence of vector bundles $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ we have $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$ hence we get a well-defined group homomorphism $\det : K_0(X) \rightarrow \text{Pic}(X)$.

The above diagram commutes because the construction of the determinant map is natural. The surjectivity of \det also comes from the construction since we could pick \mathcal{E} to be any line bundle and hence $\det(\mathcal{E}) = \mathcal{E}$. \square

3. THE IRREDUCIBLE AND REDUCED CASE OF THE MAIN THEOREM

To illustrate the idea, we focus on the case that X is an irreducible, reduced, projective curve over k in this section.

In this case let $p : \tilde{X} \rightarrow X$ be a (geometric) resolution of singularity and we can obtain more information on $\text{Pic}(\tilde{X})$. First we have

Theorem 3.1 ([6] Corollary 7.4.41). *Let \tilde{X} be a smooth, connected, projective curve over an algebraically closed field k , of genus g . Let $\text{Pic}^0(\tilde{X})$ denote the subgroup of $\text{Pic}(\tilde{X})$ consisting of divisors of degree 0. Let $n \in \mathbb{Z}$ be non-zero and $\text{Pic}^0(\tilde{X})[n]$ denote the kernel of the multiplication by n map.*

- (1) *If $(n, \text{char}(k)) = 1$, then $\text{Pic}^0(\tilde{X})[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$;*
- (2) *If $p = \text{char}(k) > 0$, then there exists an $0 \leq h \leq g$ such that for any $n = p^m$, we have $\text{Pic}^0(\tilde{X})[n] = (\mathbb{Z}/n\mathbb{Z})^h$.*

Corollary 3.2. *Let \tilde{X} be a smooth, connected, projective curve over an algebraically closed field k of genus $g \geq 1$, then $\text{Pic}^0(\tilde{X})$ and hence $\text{Pic}(\tilde{X})$ are not finitely generated as an abelian group. Moreover, for any non-zero integer n , $n\text{Pic}(\tilde{X})$ is not finitely generated.*

Proof. It is an immediate consequence of Theorem 3.1. \square

Remark 7. If the base field k is not algebraically closed, then $\text{Pic}^0(\tilde{X})$ may be finitely generated. For example if $k = \mathbb{Q}$ and \tilde{X} is a smooth elliptic curve, then by Mordell theorem, $\text{Pic}^0(\tilde{X})$ is a finitely generated abelian group.

Let Z be the closed subset consisting of singular points of X and $U = X - Z$. Since $p : \tilde{X} \rightarrow X$ is a resolution of singularity, the restriction of p

$$p|_{p^{-1}(U)} : p^{-1}(U) \xrightarrow{\cong} U$$

is an isomorphism.

We want to understand the picard group of U . In fact we have the following result

Lemma 3.3. *Let \tilde{X} be a smooth and connected projective curve with genus $g \geq 1$ over an algebraically closed field k . Let U be a non-empty open subset of \tilde{X} . Then $\text{Pic}(U)$ is not finitely generated. Moreover, for any non-zero integer n , $n\text{Pic}(U)$ is not finitely generated.*

Proof. This is actually part of [6] Exercise 7.4.9. Thanks to Georges Elencwajg for helping with the proof. Actually we can write $U = \tilde{X} \setminus \{p_1, \dots, p_l\}$. It follows that the kernel of the natural homomorphism $\text{Pic}^0(\tilde{X}) \rightarrow \text{Pic}^0(U)$ is the subgroup of $\text{Pic}^0(\tilde{X})$ generated by $[p_i] - [p_j]$, hence is finitely generated. Then this lemma is a consequence of Corollary 3.2. \square

It is also necessary to know the relation between the Picard group of a non-smooth curve X and its non-empty subscheme U , which is given in the following lemma.

Lemma 3.4. *Let X be a (not necessarily smooth) curve over an algebraically closed field k . Let U be an open subscheme of X .*

Let \mathcal{L} be a line bundle on U . Then we can always extend \mathcal{L} to a line bundle on X . As a result, the restriction map of the Picard groups

$$r : \text{Pic}(X) \rightarrow \text{Pic}(U)$$

is surjective

Proof. One way to proof this result (thanks to Kęstutis Česnavičius for pointing it out) is to first find a Cartier divisor D on U whose associated line bundle is \mathcal{L} . The existence of such D is guaranteed by [2] Proposition 21.3.4 (a). Then apply [2] Proposition 21.9.4 we can extend D to a Cartier divisor D' on X , whose associated line bundle \mathcal{L}' gives an extension of \mathcal{L} . \square

The next Proposition is the key step of our proof.

Proposition 3.5. *Let X be a reduced, irreducible, projective curve of geometric genus $g \geq 1$ over an algebraically closed field k , then the image of the Cartan homomorphism*

$$c : K_0(X) \rightarrow G_0(X)$$

is not finitely generated.

Proof. First let Z be the closed subset consisting of singular points of X and $U = X - Z$ be the smooth open subscheme. We have the restriction maps $r : K_0(X) \rightarrow K_0(U)$ and $r : G_0(X) \rightarrow G_0(U)$ and they give the commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{c} & G_0(X) \\ \downarrow r & & \downarrow r \\ K_0(U) & \xrightarrow{c} & G_0(U) \end{array}$$

Since U is smooth, by Proposition 2.3 the bottom map is an isomorphism.

Now assume the image of the top map is finitely generated, then the image of the composition $r \circ c : K_0(X) \rightarrow G_0(U)$ is also finitely generated. Since we have the isomorphism $c : K_0(U) \xrightarrow{\cong} G_0(U)$, the left vertical map $r : K_0(X) \rightarrow K_0(U)$ must also have finitely generated image. Therefore the image of the composition

$$K_0(X) \xrightarrow{r} K_0(U) \xrightarrow{\det} \text{Pic}(U)$$

is finitely generated.

On the other hand we consider the commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\det} & \text{Pic}(X) \\ \downarrow r & & \downarrow r \\ K_0(U) & \xrightarrow{\det} & \text{Pic}(U) \end{array}$$

By Proposition 2.4 and Lemma 3.4, the top and the right vertical map of the above diagram are surjective and so does their composition. As a result $\text{Pic}(U) = \text{Pic}(p^{-1}(U))$ is finitely generated, which is contradictory to Lemma 3.3. \square

Corollary 3.6. *Let X be a reduced, irreducible, projective curves of geometric genus $g \geq 1$ over an algebraically closed field k . If the inclusion $D^{\text{perf}}(X) \rightarrow D^b(\text{coh}(X))$ factors through a triangulated category \mathcal{S} , then \mathcal{S} cannot have a full exceptional collection.*

Proof. The composition

$$K_0(X) \rightarrow K_0(\mathcal{S}) \rightarrow G_0(X)$$

coincides with the Cartan homomorphism $c : K_0(X) \rightarrow G_0(X)$. By Proposition 3.5, the image of the Cartan homomorphism is not finitely generated, hence $K_0(\mathcal{S})$ is not finitely generated. Then by Proposition 2.1, \mathcal{S} cannot have a full exceptional collection. \square

Corollary 3.7. *Let X be a reduced, irreducible, projective curves of geometric genus $g \geq 1$ over an algebraically closed field k . Let $(\mathcal{T}, \pi_*, \pi^*)$ be a categorial resolution of X . Then \mathcal{T}^c cannot have a full exceptional collection.*

Proof. By the definition of categorical resolution, the composition

$$D^{\text{perf}}(X) \xrightarrow{\pi^*} \mathcal{T}^c \xrightarrow{\pi_*} D^b(\text{coh}(X))$$

is the same as the inclusion $D^{\text{perf}}(X) \hookrightarrow D^b(\text{coh}(X))$. Therefore the composition

$$K_0(X) \rightarrow K_0(\mathcal{T}^c) \rightarrow G_0(X)$$

coincides with the Cartan homomorphism $c : K_0(X) \rightarrow G_0(X)$. Then it is a direct consequence of Corollary 3.6. \square

4. THE GENERAL CASE OF THE MAIN THEOREM

In this section we consider the case that X is not irreducible nor reduced. In this case we still want to show that the image of the Cartan homomorphism $c : K_0(X) \rightarrow G_0(X)$ is not finitely generated but the proof is more involved.

Let X_{red} denote the associated reduced scheme of X and $i : X_{\text{red}} \rightarrow X$ the natural closed immersion. Then X_{red} is a reduced, projective curve with the same geometric genus as X .

First we investigate the $g = 0$ case, which is the following Proposition.

Proposition 4.1. *Let X be a projective curve over an algebraically closed field k of geometric genus $g = 0$, then X has a categorical resolution $(\mathcal{T}, \pi^*, \pi_*)$ such that \mathcal{T}^c has a full exceptional collection.*

Proof. As we mentioned in the Introduction, the result in this Proposition is a direct consequence of the construction of categorical resolution in [5], although it is not explicitly stated in [5].

First [5] Equation (59) in page 69 gives a chain

$$\begin{array}{ccccccc} X_m & \longrightarrow & X_{m-1} & \longrightarrow & \dots & \longrightarrow & X_1 \longrightarrow X_0 \equiv X \\ & & \uparrow & & & & \uparrow \\ & & Z_{m-1} & & & & Z_1 \\ & & & & & & \uparrow \\ & & & & & & Z_0 \end{array} \quad (1)$$

where each X_{i+1} is the blowup of X_i at the center Z_i and $(X_m)_{\text{red}}$ is smooth.

Moreover [5] Equation (61) in page 71 tells us that there exists a categorical resolution \mathcal{T} of X such that its subcategory \mathcal{T}^c has the following semiorthogonal decomposition

$$\begin{aligned} \mathcal{T}^c = & \underbrace{\langle D^b(\text{coh}(Z_0)) \dots D^b(\text{coh}(Z_0)) \rangle}_{n_0 \text{ times}}, \\ & \underbrace{\langle D^b(\text{coh}(Z_{m-1})) \dots D^b(\text{coh}(Z_{m-1})) \rangle}_{n_{m-1} \text{ times}}, \\ & \underbrace{\langle D^b(\text{coh}((X_m)_{\text{red}})) \dots D^b(\text{coh}((X_m)_{\text{red}})) \rangle}_{n_m \text{ times}} \end{aligned} \quad (2)$$

where the n_i 's are certain multiples given in [5] after Equation (61) and we do not need their precise definition.

Since X is of dimension 1, each of the Z_i is 0-dimensional hence $D^b(\text{coh}(Z_i))$ has a full exceptional collection. Moreover since X is of genus 0, we have $(X_m)_{\text{red}}$ is a finite product of \mathbb{P}^1 's hence $D^b(\text{coh}((X_m)_{\text{red}}))$ also has a full exceptional collection. As a result \mathcal{T}^c has a full exceptional collection. \square

Then we consider the $g \geq 1$ case. By Definition 2.3 and 2.4 we have the natural map

$$i^* : K_0(X) \rightarrow K_0(X_{\text{red}})$$

and

$$i_* : G_0(X_{\text{red}}) \rightarrow G_0(X).$$

For i_* we have the following "devissage" theorem.

Theorem 4.2. *[9] Chapter II Corollary 6.3.2] Let X be a Noetherian scheme, and X_{red} the associated reduced scheme. Then $i_* : G_0(X_{\text{red}}) \rightarrow G_0(X)$ is an isomorphism.*

Proof. See [9] Chapter II Corollary 6.3.2. \square

However, the following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{c} & G_0(X) \\ \downarrow i^* & & \cong \uparrow i_* \\ K_0(X_{\text{red}}) & \xrightarrow{c} & G_0(X_{\text{red}}) \end{array}$$

does not commute. Hence we cannot directly apply the result in Section 3 and need to find another way.

Let $X = \cup_{i=1}^m X_i$ be the decomposition into irreducible components, hence $X_{\text{red}} = \cup_{i=1}^m (X_i)_{\text{red}}$ (Do not confused with the X_i 's in the proof of Proposition 4.1). Since X has geometric genus ≥ 1 , at least one of the irreducible components X_i 's also has geometric genus ≥ 1 , say X_1 .

For an non-empty, open, irreducible subscheme U of X_1 we also consider U_{red} . We can make U small enough so that both U and U_{red} are affine and U_{red} is smooth. Let $U = \text{Spec}(A)$ and $U_{\text{red}} = \text{Spec}(A/I)$ where I is the nilpotent radical of A with $I^{l+1} = 0$. Since U is irreducible, I is also the minimal prime ideal of A . Let \mathcal{I} denote the associated sheaf on U .

Let us consider the diagram

$$\begin{array}{ccc} K_0(U) & \xrightarrow{c} & G_0(U) \\ \downarrow i^* & & \uparrow i_* \\ K_0(U_{\text{red}}) & \xrightarrow{c} & G_0(U_{\text{red}}) \end{array}$$

Again it does not commute. Nevertheless we will prove that it is not too far from commutative.

First let us fix the notations. Let e_U denote the element $[\mathcal{O}_U]$ in $G_0(U)$ and $e_{U_{\text{red}}}$ denote the element $[\mathcal{O}_{U_{\text{red}}}]$ in $G_0(U_{\text{red}})$.

Lemma 4.3. *We can choose U small enough such that there is a non-zero integer n such that*

$$e_U = n i_*(e_{U_{\text{red}}}).$$

Proof. By Theorem 4.2, i_* is an isomorphism so it is sufficient to prove

$$i_*^{-1}(e_U) = n e_{U_{\text{red}}}$$

in $G_0(U_{\text{red}})$.

It is clear that in $G_0(U_{\text{red}})$ we have

$$i_*^{-1}(e_U) = e_{U_{\text{red}}} + [\mathcal{I}/\mathcal{I}^2] + \dots + [\mathcal{I}^{l-1}/\mathcal{I}^l] + [\mathcal{I}^l]. \quad (3)$$

Each of the $\mathcal{I}^{k-1}/\mathcal{I}^k$ is a coherent sheaf on the smooth scheme U_{red} hence we have a resolution of finite length

$$0 \rightarrow \mathcal{P}_k^{m_k} \rightarrow \mathcal{P}_k^{m_k-1} \rightarrow \dots \rightarrow \mathcal{P}_k^0 \rightarrow \mathcal{I}^{k-1}/\mathcal{I}^k \text{ for } 1 \leq k \leq l+1.$$

where the $\mathcal{P}_k^{m_k-j}$'s are locally free sheaves on U_{red} . We can shrink U further to make all the $\mathcal{P}_k^{m_k-j}$'s are free sheaves on U_{red} . Hence for each k there is an integer n_k such that

$$[\mathcal{I}^{k-1}/\mathcal{I}^k] = n_k e_{U_{\text{red}}}$$

and as a result there is an integer n such that

$$i_*^{-1}(e_U) = n e_{U_{\text{red}}}$$

in $G_0(U_{\text{red}})$.

We still need to show that $n \neq 0$. This can be achieved by localizing to the generic point of U . Recall that I is the minimal prime ideal of A hence I corresponds to the generic point of U .

Let us denote A_I , the localization of A at I by B and denote the ideal IB by J . Moreover we denote $\text{Spec}(B)$ by V and similarly denote $\text{Spec}(B/J)$ by V_{red} . Let $f : V \rightarrow U$, $f_{\text{red}} : V_{\text{red}} \rightarrow U_{\text{red}}$, and $j : V_{\text{red}} \rightarrow V$ be the natural maps.

Since $f : V \rightarrow U$ is flat, we can define the pull-back map $f^* : G_0(U) \rightarrow G_0(V)$.

Let us denote the class $[\mathcal{O}_V]$ in $G_0(V)$ by e_V . By definition $f^*(e_U) = e_V$. If $e_U = 0$ then we have $e_V = 0$ and $j_*^{-1}(e_V) = 0$.

On the other hand $B/J = A_I/I_I \cong \text{Frac}(A/I)$ is a field hence $G_0(V_{\text{red}}) = G_0(B/J) \cong \mathbb{Z}$. Similar to Equation (3) we have

$$j_*^{-1}(e_V) = [B/J] + [J/J^2] + \dots + [J^{l-1}/J^l] + [J^l].$$

Each of the J^{m-1}/J^m is a vector space over the field B/J hence the right hand side cannot be zero in $G_0(V_{\text{red}})$. \square

Proposition 4.4. *Let U and n be as in Lemma 4.3. Then for any element $a \in K_0(U)$ we have*

$$c(a) = n i_* c i^*(a),$$

i.e. the diagram

$$\begin{array}{ccc} K_0(U) & \xrightarrow{c} & G_0(U) \\ \downarrow n i^* & & \cong \uparrow i_* \\ K_0(U_{\text{red}}) & \xrightarrow{c} & G_0(U_{\text{red}}) \end{array}$$

commutes.

Proof. We need the following lemma.

Lemma 4.5. *For any Noetherian scheme U , $G_0(U_{\text{red}})$ has a $K_0(U)$ -module structure. Moreover, the map $i_* : G_0(U_{\text{red}}) \rightarrow G_0(U)$ is a morphism of $K_0(U)$ -modules.*

Proof of Lemma 4.5. First the $K_0(U)$ -module structure on $G_0(U_{\text{red}})$ is given by composing with i^* . More explicitly, for $a \in K_0(U)$ and $m \in G_0(U_{\text{red}})$ we define

$$a \cdot m = i^*(a) \cdot m$$

where the right hand side uses the $K_0(U_{\text{red}})$ -module structure on $G_0(U_{\text{red}})$.

Then we need to show that i_* is a $K_0(U)$ -module map, i.e.

$$i_*(i^*(a) \cdot m) = a \cdot i_*(m).$$

But this is exactly the projection formula. □

Now we can prove Proposition 4.4. Let us denote $[\mathcal{O}_U] \in K_0(U)$ by 1_U and $[\mathcal{O}_{U_{\text{red}}}] \in K_0(U_{\text{red}})$ by $1_{U_{\text{red}}}$. Then it is clear that

$$c(1_U) = e_U \text{ and } c(1_{U_{\text{red}}}) = e_{U_{\text{red}}}.$$

Then for any $a \in K_0(U)$ we have

$$\begin{aligned} c(a) &= c(a \cdot 1_U) \\ &= a \cdot e_U \\ &= a \cdot (n i_*(e_{U_{\text{red}}})) \text{ (Lemma 4.3)} \\ &= n(a \cdot i_*(e_{U_{\text{red}}})) \\ &= n i_*(i^*(a) \cdot e_{U_{\text{red}}}) \text{ (Lemma 4.5)} \\ &= n i_* c i^*(a). \end{aligned}$$

□

Now we are ready to prove the following Proposition, which is the general version of Proposition 3.5.

Proposition 4.6. *Let X be a projective curves of geometric genus $g \geq 1$ over an algebraically closed field k , then the image of the Cartan homomorphism*

$$c : K_0(X) \rightarrow G_0(X)$$

is not finitely generated.

Proof. First let U be as in Lemma 4.3 and Proposition 4.4. By Proposition 4.4 and Theorem 4.2 there is a non-zero integer n such that the following diagram commutes

$$\begin{array}{ccc} K_0(U) & \xrightarrow{c} & G_0(U) \\ \downarrow n i^* & & \downarrow (i_*)^{-1} \\ K_0(U_{\text{red}}) & \xrightarrow{c} & G_0(U_{\text{red}}) \end{array}$$

hence the diagram

$$\begin{array}{ccc}
K_0(X) & \xrightarrow{c} & G_0(X) \\
r \downarrow & & \downarrow r \\
K_0(U) & \xrightarrow{c} & G_0(U) \\
\downarrow n i^* & & \downarrow (i_*)^{-1} \\
K_0(U_{\text{red}}) & \xrightarrow{c} & G_0(U_{\text{red}})
\end{array}$$

commutes. For short we have

$$\begin{array}{ccc}
K_0(X) & \xrightarrow{c} & G_0(X) \\
\downarrow n i^* r & & \downarrow (i_*)^{-1} r \\
K_0(U_{\text{red}}) & \xrightarrow{c} & G_0(U_{\text{red}})
\end{array}$$

Now assume the image of $c : K_0(X) \rightarrow G_0(X)$ is finitely generated. Since U_{red} is smooth, the $c : K_0(U_{\text{red}}) \rightarrow G_0(U_{\text{red}})$ in the above diagram is an isomorphism, hence the image of $n i^* r$ is also finitely generated.

Next we observe that we have the commutative diagrams

$$\begin{array}{ccc}
K_0(X) & \xrightarrow{n i^*} & K_0(X_{\text{red}}) \\
r \downarrow & & \downarrow r \\
K_0(U) & \xrightarrow{n i^*} & G_0(U_{\text{red}})
\end{array}$$

and

$$\begin{array}{ccc}
K_0(X) & \xrightarrow{\det} & \text{Pic}(X) \\
n i^* \downarrow & & \downarrow n i^* \\
K_0(X_{\text{red}}) & \xrightarrow{\det} & \text{Pic}(X_{\text{red}}) \\
r \downarrow & & \downarrow r \\
K_0(U_{\text{red}}) & \xrightarrow{\det} & \text{Pic}(U_{\text{red}})
\end{array}$$

From the left-bottom composition of the above diagram we know that the image of $\det \circ r \circ (n i^*)$ is finitely generated.

On the other hand we will study the top-right composition of the above diagram. By Proposition 2.4 the map \det is surjective and by Lemma 3.4 the map r is also surjective. As for the map i^* we need the following lemma.

Lemma 4.7. *[6] Lemma 7.5.11] Let X be a connected projective curve over an algebraically closed field k . Then $i^* : \text{Pic}(X) \rightarrow \text{Pic}(X_{\text{red}})$ is surjective.*

Proof of Lemma 4.7. See [6] Lemma 7.5.11. □

Then it is clear that the image of $r \circ (n i^*) \circ \det$ is $n \text{Pic}(U_{\text{red}})$. Compare with the left-bottom composition we get the conclusion that $n \text{Pic}(U_{\text{red}})$ is finitely generated, which is contradictory to Lemma 3.3. □

Corollary 4.8. *Let X be a projective curves of geometric genus $g \geq 1$ over an algebraically closed field k . If the inclusion $\text{D}^{\text{perf}}(X) \rightarrow \text{D}^b(\text{coh}(X))$ factors through a triangulated category \mathcal{S} , then \mathcal{S} cannot have a full exceptional collection.*

Proof. The proof is almost the same as that of Corollary 3.6 except that we use Proposition 4.6 instead of Proposition 3.5. □

Theorem 4.9. *[See Theorem 1.1] Let X be a projective curve over an algebraically closed field k . Let $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of X . If the geometric genus of X is ≥ 1 , then \mathcal{T}^c cannot have a full exceptional collection.*

In other words, X has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of X equals to 0.

Proof. Since we have Proposition 4.1, it is sufficient to prove the first claim of the theorem, which is a direct consequence of Corollary 4.8. \square

Remark 8. In the proof we did not use the fact the \mathcal{T} is a smooth triangulated category.

Remark 9. The proof of Theorem 4.9 fails if the base field k is not algebraically closed. The main reason is when k is not algebraically closed, the picard group may be finitely generated. See Remark 7 after Corollary 3.2.

Nevertheless, we expect that the result of Theorem 4.9 is still true in the non-algebraically closed case. We believe that a proof could be achieved through a systematic study of the behavior of categorical resolution under scalar extension and we will leave this topic for a future paper.

It is worthwhile to mention that we have another application of Proposition 4.6 (thanks to Igor Burban for pointing it out).

Theorem 4.10. *Let X be a projective curve over an algebraically closed field k of geometric genus ≥ 1 . Let $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of X . Then \mathcal{T}^c cannot have a tilting object, moreover there cannot be a finite dimensional k -algebra Λ of finite global dimension such that*

$$\mathcal{T}^c \cong D^b(\Lambda - \text{mod})$$

where $D^b(\Lambda - \text{mod})$ is the derived category of bounded complexes of finitely generated Λ -modules.

Proof. With Proposition 4.6 it is sufficient to prove that the Grothendieck group $K_0(D^b(\Lambda - \text{mod}))$ is finitely generated. The proof is as follows: Since Λ is finite dimensional, it is a finitely generated Artinian k -algebra, hence every finitely generated Λ -module has a composition series. Moreover the set of isomorphic classes of simple Λ -module is finite. We get the desired result. \square

Remark 10. Again in the proof we did not use that fact that Λ is of finite global dimension, which corresponds to the smoothness of \mathcal{T} .

ACKNOWLEDGEMENT

The author first wants to thank Valery Lunts for introducing him to this topic and for very helpful comments and suggestions during this work. Igor Burban suggests the author to investigate the non-irreducible case and shares the ideas on tilting object and the author is grateful to him too. Moreover the author would like to thank Dave Anderson, Kęstutis Česnavičius, Georges Elencwajg, Volodimir Gavran, Adeel Khan, Sándor Kovács, and Jason Starr for their help on algebraic K-theory and the theory of algebraic curves.

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